

A NOTE ON MIXING TIMES OF PLANAR RANDOM WALKS

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ABSTRACT. We present an infinite family of finite planar graphs $\{X_n\}$ with degree at most five and such that for some constant $c > 0$,

$$\lambda_1(X_n) \geq c \left(\frac{\log \text{diam}(X_n)}{\text{diam}(X_n)} \right)^2,$$

where λ_1 denotes the smallest non-zero eigenvalue of the graph Laplacian. This significantly simplifies a construction of Louder and Souto.

We also remark that such a lower bound cannot hold when the diameter is replaced by the average squared distance: There exists a constant $c > 0$ such that for any family $\{X_n\}$ of planar graphs we have

$$\lambda_1(X_n) \leq c \left(\frac{1}{|X_n|^2} \sum_{x,y \in X_n} d(x,y)^2 \right)^{-1},$$

where d denotes the path metric on X_n .

Recently, Louder and Souto [3] answered negatively a question of Benjamini and Curien [1] by showing that there is an infinite family of bounded-degree planar graphs for which the mixing time is asymptotically less than the square of the diameter. Their construction uses expander graphs to construct a surface which they triangulate to arrive at a planar graph. We present a simple family of graphs exhibiting the same result.

1. THE LAPLACIAN

For a finite, undirected graph $G = (V, E)$ with edge and vertex weights $w : E \rightarrow [0, \infty)$ and $\pi : V \rightarrow (0, \infty)$, one defines the combinatorial Laplacian $L : \ell^2(V, \pi) \rightarrow \ell^2(V, \pi)$ by

$$Lf(x) = \sum_{y: \{x,y\} \in E} \frac{w(x,y)}{\pi(x)} (f(x) - f(y)).$$

Here, $\ell^2(V, \pi)$ is equipped with the inner product $\langle f, g \rangle_{\ell^2(V, \pi)} = \sum_{x \in V} \pi(x) f(x) g(x)$. This coincides with the unweighted combinatorial Laplacian when $w \equiv 1$ and $\pi \equiv 1$.

Recall that the smallest non-zero eigenvalue of L as an operator on $\ell^2(V, \pi)$ is given by

$$\lambda_1(G) = \min_{f \in \ell^2(V, \pi)} \left\{ \frac{\sum_{\{x,y\} \in E} w(x,y) (f(x) - f(y))^2}{\sum_{x \in V} \pi(x) f(x)^2} : \sum_{x \in V} \pi(x) f(x) = 0 \right\}.$$

The Cheeger constant of G is defined by

$$h(G) = \min_{S \subseteq V} \left\{ \frac{\sum_{\{x,y\} \in E} w(x,y) |\mathbf{1}_S(x) - \mathbf{1}_S(y)|}{\pi(S)} : \pi(S) \leq \pi(V)/2 \right\},$$

where $\mathbf{1}_S$ denotes the characteristic function of S and $\pi(S) = \sum_{v \in V} \pi(v)$ for a subset $S \subseteq V$.

J.L. partially supported by NSF CCF-0915251 and a Sloan Research Fellowship. This work was completed during a visit of the authors to Microsoft Research.

We recall the discrete Cheeger inequality

$$\lambda_1(G) \geq \frac{h(G)^2}{2 d_{\max}}, \quad (1)$$

where $d_{\max} = \max_{x \in V} \left(\pi(x)^{-1} \sum_{y: \{x,y\} \in E} w(x,y) \right)$ is the maximum “degree” in G .

For a graph G , we use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G , respectively. Unless otherwise stated, the edge and vertex weights on a graph G are taken to be uniform.

2. THE CONSTRUCTION

Let T_h denote the complete, rooted binary tree of height h . Let $T_{h,k}$ denote the result of subdividing every edge of T_h by k . If $T_{h,k}$ has root r , we write

$$V_\ell = \{v \in V(T_{h,k}) : \text{dist}(v, r) = \ell\}$$

for the set of nodes at distance ℓ from r . Fix an in-order traversal of $T_{h,k}$. For each $\ell = 0, 1, \dots, hk$, we add a path P_ℓ on the nodes at depth ℓ . Specifically, edges go between consecutive nodes of V_ℓ in the in-order traversal. Call this final graph $\hat{T}_{h,k}$. It is straightforward to verify that $\hat{T}_{h,k}$ is planar.

Theorem 1. *For every $h \geq 1$ and $k = 2^h$, the following bounds hold:*

- i) $\text{diam}(\hat{T}_{h,k}) \geq hk$, and
- ii) $\lambda_1(\hat{T}_{h,k}) \geq \frac{1}{7k^2} \geq \left(\frac{\log_2 \text{diam}(\hat{T}_{h,k})}{6 \text{diam}(\hat{T}_{h,k})} \right)^2$.

Proof. The diameter bound is (i) clear, so we focus on (ii).

Let $V = V(\hat{T}_{h,k})$. Consider any $f : V \rightarrow \mathbb{R}$ with $\sum_{x \in V} f(x) = 0$. Define $\bar{f} : V \rightarrow \mathbb{R}$ as follows: For $x \in V_\ell$,

$$\bar{f}(x) = \frac{1}{|V_\ell|} \sum_{x \in V_\ell} f(x).$$

Observe that $\sum_{x \in V} \bar{f}(x) = 0$ holds as well.

Now, from (1) applied to the graph P_ℓ , which has $h(P_\ell) \geq 2/|V_\ell|$, we have

$$\sum_{\{x,y\} \in P_\ell} (f(x) - f(y))^2 \geq \frac{1}{|V_\ell|^2} \sum_{x \in V_\ell} (f(x) - \bar{f}(x))^2.$$

Using $|V_\ell| \leq 2^h$ and summing over all ℓ yields

$$\sum_{\ell=1}^h \sum_{\{x,y\} \in E(P_\ell)} (f(x) - f(y))^2 \geq 2^{-2h} \|f - \bar{f}\|^2 \geq \frac{1}{k^2} \|f - \bar{f}\|^2. \quad (2)$$

That covers the “horizontal” edges of $\hat{T}_{h,k}$. We claim that the following bound holds for the “vertical” edges:

$$\sum_{\{x,y\} \in E(T_{h,k})} (\bar{f}(x) - \bar{f}(y))^2 \geq \frac{1}{6k^2} \|\bar{f}\|^2. \quad (3)$$

Since $\sum_{x \in V} \bar{f}(x) = 0$ and \bar{f} is constant on the level sets $\{V_\ell\}$, this is implied by a lower bound on the spectral gap of the weighted quotient graph $Q_{h,k}$ of $\hat{T}_{h,k}$, where each level set V_ℓ is identified to a single vertex of weight $|V_\ell|$.

But $Q_{h,k}$ is simply a k -subdivision of the weighted graph Q_h which has vertex set $\{0, 1, \dots, h\}$, edge weights $w(j, j+1) = 2^{j+1}$, and vertex weights $\pi(j) = 2^j$. In particular, $\lambda_1(Q_{h,k}) = \frac{1}{k^2} \lambda_1(Q_h)$. Hence (3) is implied by the lower bound

$$\lambda_1(Q_h) = \min_{g: \{0,1,\dots,h\} \rightarrow \mathbb{R}} \left\{ \frac{\sum_{j=0}^{h-1} 2^{j+1} (g(j) - g(j+1))^2}{\sum_{j=0}^h 2^j g(j)^2} : \sum_{j=0}^h 2^j g(j) = 0 \right\} \geq \frac{1}{6}.$$

But this follows from (1), because $h(Q_h) \geq 1$, which is easily verified: Any S of weight at most half cannot contain h , which has $\pi(h) > \pi(V(Q_h))/2$. Thus for any such $S \subseteq \{0, 1, \dots, h-1\}$, if $j = \max(S)$, then $\pi(S) \leq 2^j + 2^{j-1} + \dots + 1 \leq 2^{j+1}$, while the edge $(j, j+1)$ leaves S and has $w(j, j+1) = 2^{j+1}$.

Finally, we claim that

$$\sum_{\{x,y\} \in E(T_{h,k})} (f(x) - f(y))^2 \geq \sum_{\{x,y\} \in E(T_{h,k})} (\bar{f}(x) - \bar{f}(y))^2. \quad (4)$$

This follows by applying Jensen's inequality to the edges from V_ℓ to $V_{\ell+1}$ for each value of ℓ . Now summing lines (2) and (4) and using (3) yields

$$\sum_{\{x,y\} \in E(\widehat{T}_{h,k})} (f(x) - f(y))^2 \geq \frac{1}{k^2} \left(\frac{\|\bar{f}\|^2}{6} + \|f - \bar{f}\|^2 \right) \geq \frac{1}{7k^2} \|f\|^2,$$

completing the proof. \square

3. CONCLUSION

As Yuval Peres pointed out to us, one can check that the mixing time of simple random walk on $\widehat{T}_{h,k}$ is on the order of hk^2 for $k = 2^h$, which is a $\log |V(\widehat{T}_{h,k})|$ factor larger than the relaxation time (i.e., the inverse spectral gap). Observe that although the diameter of $\widehat{T}_{h,k}$ is hk , the average distance between a uniformly random pair of vertices is bounded by $O(k)$. And indeed, it is true in general that in a bounded-degree planar graph, the mixing time is at least the average of the squared distance.

Theorem 2 ([2]). *For some constant $c > 0$, the following holds. Let $G = (V, E)$ be a planar graph with path metric d . Then,*

$$\lambda_1(G) \leq c \left(\frac{1}{|V|^2} \sum_{x,y \in V} d(x,y)^2 \right)^{-1}.$$

Proof. By [2, Thm 4.4], there exists a universal constant $C > 0$ and a 1-Lipschitz mapping $f : V \rightarrow \mathbb{R}$ such that

$$\sum_{x,y \in V} |f(x) - f(y)|^2 \geq C \sum_{x,y \in V} d(x,y)^2.$$

By a standard calculation using the fact that $|E| \leq 3|V|$, one has

$$\lambda_1(G) \leq \frac{\sum_{\{x,y\} \in E} |f(x) - f(y)|^2}{\frac{1}{2|V|} \sum_{x,y \in V} |f(x) - f(y)|^2} \leq \frac{(3|V|)(2|V|)}{C \sum_{x,y \in V} d(x,y)^2},$$

completing the proof. \square

We remark that a similar bound holds for any graph of bounded genus, or more generally, for any graph excluding a fixed minor; see [2]. For families of graphs of unbounded degree, one should

consider the normalized Laplacian $\mathcal{L}f(x) = \sum_{y:\{x,y\}\in E} (\pi(x)\pi(y))^{-1/2} (f(x) - f(y))$, where $\pi(x)$ denotes the stationary probability of $x \in V$. A similar argument shows that for some $c > 0$,

$$\lambda_1(\mathcal{L}) \leq c \left(\sum_{x,y \in V} \pi(x)\pi(y)d(x,y)^2 \right)^{-1},$$

implying that, in planar graphs, the mixing time is at least the average squared distance when points are chosen uniformly from the stationary measure.

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